Semi-discrete time-domain sensitivity analysis of electromagnetic field
K.M. Gawrylczyk, M. Kugler

Department of Electrical and Computer Engineering
Szczecin University of Technology
Sikorskiego 37, 70-313 Szczecin, Poland
E-mail: kmg@ps.pl, mkugler@ps.pl

Abstract: Adjoint models in sensitivity analysis [1] utilize the unit step current excitation. In our earlier works, the FE-time-stepping scheme was exploited. The proposed semi-discrete method allows us to obtain time-domain solution without time-stepping. For space discretization we use finite elements, as usual. The semi-discrete method allows us to determine analytic and continuous solution for any given time of analysis. In this work we consider only axial-symmetric models containing linear and isotropic media.

Keywords: transient electromagnetic fields, finite element method, sensitivity analysis.

I INTRODUCTION

The sensitivity analysis has been developed for inverse tasks, as structure recognition, or optimisation of electric devices. The inverse task bases on sequential forward-solutions, while the optimised parameters change according to chosen optimisation method. We use the Gauss-Newton algorithm, that is we minimize the form

\[
\min_{\Delta \xi} \| \Delta A \| - [S] \frac{\partial \xi}{\partial A},
\]

where \( \Delta A \) means the difference between desired and simulated potential values, \( \Delta \xi \) – parameter corrections and \( S \) – sensitivity matrix. The matrix \( S \) may be non-quadratic, than we use singular value decomposition. The sensitivity may be calculated versus either material parameters, as electric conductivity in finite elements, or geometric parameters, describing position of optimized boundary. If the excitation has the form of impulse, the consideration in time-domain are necessary. The advantage of the method basing on transient course versus harmonic one lies in large number of input data received for specific time moments. The recognition of conductivity distribution utilizing Finite Element Time-Domain (FETD) and over-determined equation system one can find in [1].

The sensitivity equation used to obtain the matrix \( S \) for the case of electric intensity vector sensitivity versus electric conductivity \( \gamma \) takes the form of

\[
\int_0^T \int_0^\Omega J^+ (t) \partial E (t) d\Omega dt = \int_0^T \int_0^\Omega \partial \gamma \cdot E^+ (t) E(t) d\Omega dt.
\]

The integration follows for forward time \( t \) and the backward time \( \tau \). The both times are combined together, as shown in Fig.1.

Figure 1: Forward time \( t \) and the backward time \( \tau \).

To hold the time stepping according to Fig.1 it isn’t possible to apply the adaptive time-step and our earlier algorithms worked with constant time step. Evaluating the sensitivity for large \( T (\tau = T - t) \) we had to carry out more than 1000 steps, and despite of using LU decomposition the calculation effort was very large. This is the reason, why we propose the semi-discrete method.
II SEMI-DISCRETE FINITE ELEMENT ANALYSIS

The diffusion equation we are solving has the form of

$$[K][A(t)] + [M]\frac{\partial A(t)}{\partial t} = \{i(t)\}. \quad (3)$$

with: $K$ – stiffness and $M$ – mass matrix. $A$ means magnetic vector potential, if need modified for the case of cylindrical symmetry. Excitation current $i(t)$ of adjoint model has the form of unit step $[1]$. Solving (3) as usual transient equation with zero initial conditions we obtain the semi-discrete solution:

$$\{A(t)\} = \left([1] - \exp\left(-t[M]^{-1}[K]\right)\right) \cdot [K]^{-1} \cdot \{i\} \quad (4)$$

While the inversion of mass matrix is necessary, we can’t apply (4) for the whole region. It has to be subdivided on conductive part “1” and non-conductive part “2”:

$$\begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{bmatrix} A_1(t) \\ A_2(t) \end{bmatrix} + \begin{bmatrix} M_{11} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial t} A_1(t) \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ i_s \end{bmatrix}. \quad (5)$$

We assume, that the excitation currents are located only in non-conducting part of the region. Then we obtain following matrix equation for conducting region:

$$\left([K_{11}] - [K_{12}] [K_{22}]^{-1} [K_{21}]\right) \{A_1(t)\} + [M_{11}] \{\partial A_1(t)/\partial t\} = -[K_{12}] [K_{22}]^{-1} \{i_s(t)\}, \quad \text{or:}$$

$$\begin{bmatrix} K_1 \\ K_2 \end{bmatrix} \{A_1(t)\} + \begin{bmatrix} M_{11} & 0 \\ 0 & 0 \end{bmatrix} \{\partial A_1(t)/\partial t\} = \{i_s(t)\}. \quad (6)$$

The similarity of (6) to (3) allows us to exploit the solution (4). Differentiating it, we become analytic form for electric intensity vector $E$ in original and adjoint system (*):

$$\{E_1(t)\} = \frac{\partial}{\partial t} \{A_1(t)\} = [M_{11}]^{-1} \exp\left(-t[M_1][M_{11}]^{-1}\right) \cdot \{i_1\}$$

$$\{E_1^*(\tau)\} = \frac{\partial}{\partial \tau} \{A_1^*(\tau)\} = [M_{11}]^{-1} \exp\left(-\tau[M_1][M_{11}]^{-1}\right) \cdot \{i_1^*\}, \quad \tau + t = \tau. \quad (7)$$

The comparison of electric intensity vector $E$ calculated in node “1” with FE-TS (time-stepping, 100 steps) method and semi-discrete method is shown in Fig.2 and Fig.3. The test models used for comparison are described below. We can see the good agreement of both methods.

Figure 2: Electric intensity vector for 1D test case. Figure 3: Electric intensity vector for 2D test case.
III DESCRIPTION OF TEST MODELS

For the purpose of testing a very simple model, owing cartesian symmetry, was chosen. The model consists of conducting region with material parameters $\frac{1}{6} \gamma \mu_0 = 1$ and $\mu_r = 1$. The excitation is produced by line current of the density $1\text{A/m}$. To obtain the same field strength in both models, 1D-model is driven with the current $1\text{A}$ directed to the node “11” and the 2D-model with currents of $0.5\text{A}$ directed to the nodes “21” and “22”. The element matrices for 1D-case are [4]:

$$K_e = \frac{l}{\mu_r} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \quad M_e = \frac{l}{6} \gamma \mu_0 \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \quad f_e = \frac{l}{2} \mu_0 \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

and for the 2D-case:

$$K_e = \frac{1}{4\Delta^* \mu_r} \begin{bmatrix} b_i^2 + c_i^2 & b_i b_j + c_i c_j \\ b_i b_j + c_i c_j & b_j^2 + c_j^2 \\ b_i b_j + c_i c_j & b_j b_k + c_j c_k \\ b_j b_k + c_j c_k & b_k^2 + c_k^2 \end{bmatrix}, \quad M_e = \frac{\mu_0 \gamma \Delta^*}{12} \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}, \quad f_e = \Delta^* \frac{\mu_0}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},$$

with: $b_i = y_j - y_k$, $c_i = x_k - x_j$, and $\Delta^*$ - area of the element.

Figure 4: Test models 1D and 2D.

IV SEMI-DISCRETE SENSITIVITY ANALYSIS

In further consideration we are interested for calculation of sensitivity term on the right hand side of (2):

$$\int_0^\gamma \int_\Omega \mathbf{E}^\top (\tau) \mathbf{E} (\tau) \, d\Omega \, d\tau ,$$

The integration of electric intensity vectors product over finite element of first order requires the Gauss quadrature of second order. While the Gauss-points lie on the borders of element, the combinations of nodal values products should be used. To obtain all necessary products we define the following matrix $R$:

$$R = \Delta \int_0^\gamma \int_\Omega \mathbf{E}^\top (\tau) \mathbf{E} (\tau) \, d\tau = \int_0^\gamma \left[ \mathbf{M}_i \right] \mathbf{E}^\top (\tau) \exp \left( -\tau \mathbf{K}_e \right) \left[ \mathbf{M}_i \right] \mathbf{E} (\tau) \mathbf{d} \tau = \left[ \mathbf{M}_i \right] \mathbf{E}^\top (\tau) \exp \left( -\tau \mathbf{K}_e \right) \left[ \mathbf{M}_i \right] \mathbf{E} (\tau) \mathbf{d} \tau,$$

where $\Delta$ denotes the size of finite element. The matrix $R$ can be rewritten as follows

$$R = \Delta \int_0^\gamma \left[ \mathbf{M}_i \right] \mathbf{E}^\top (\tau) \exp \left( -\tau \mathbf{K}_e \right) \left[ \mathbf{M}_i \right] \mathbf{E} (\tau) \mathbf{d} \tau,$$

where $\mathbf{d} \tau$ denotes the size of finite element.
because the term $[M_{11}]^{-1} \exp(-t[K_c][M_{11}]^{-1})$ is symmetric. The drawback of established definition is, that $R$ is singular (it contains singular term $[I_c] = \{i_c\} \cdot [i_c]^T$). For better numerical efficiency the analytical integration would be essential. Since simple disentangling of exponential functions in (12) is impossible, we propose for this aim two following methods.

**The Zassenhaus formula**

The Zassenhaus formula [6] is a version of Baker-Campbell-Hausdorff formula. It can be written involving commutator nestings:

$$\exp(X) \cdot Y - \exp(-X) = Y + [X,Y] + \frac{1}{2!} [X,[X,Y]] + \frac{1}{3!} [X,[X,[X,Y]]] + \ldots,$$

where $[X,Y] = X \cdot Y - Y \cdot X$. (13)

In our case:

$$X = -t[K_c][M_{11}]^{-1} \quad \text{and} \quad Y = \{i_c\} \cdot [i_c]^T \cdot [M_{11}]^{-1} = [I_c] \cdot [M_{11}]^{-1}. \quad \text{(14)}$$

If $1/M$ is the norm of the matrices $X$ and $Y$, the error of Zassenhaus approximation is of order $O(1/M^2)$. It means, that the sensitivity calculations basing on Zassenhaus formula converge only for small times $t$, dependent from the norm of the matrix $[K_c][M_{11}]^{-1}$. The following pictures show the results obtained with this formula. In Fig.5 one can see sensitivity term for node “1” in 1D model. The Zassenhaus formula converges for $t < 0.04s$, it means practically only in the beginning of transient processes. For the node “6”, situated nearby the surface of conductor, it converges a little bit better (Fig.6), but still not satisfying.

**Improvement of matrices commutation**

Let us take the Zassenhaus formula for two good commuting matrices. The result is then

$$\exp(X) \cdot Y \cdot \exp(-X) = Y,$$

assuming $[X,Y] = 0$. (15)

The identity matrix commutes well with all other matrices. We add and subtract identity matrix $[1]$ multiplied by an arbitrary large coefficient $C$:

$$R = A \cdot \int_{0}^{t} [M_{11}]^{-1} \exp(-t[K_c][M_{11}]^{-1}) \cdot \{I_c\} \cdot [M_{11}]^{-1} + C \cdot [1] - C \cdot [1] \cdot \exp(-t[K_c][M_{11}]^{-1}) \cdot dt. \quad \text{(16)}$$

Then, we divide our integral among two components:
\[ R = \Delta \cdot \int_{0}^{\tau} [M_{ii}]^{-1} \exp(-\tau[K_i][M_{ii}]^{-1}) \cdot \exp(\tau[LA]) \cdot \exp(-\tau[K_i][M_{ii}]^{-1}) d\tau - \]
\[ \Delta \cdot \int_{0}^{\tau} [M_{ii}]^{-1} \exp(-\tau[K_i][M_{ii}]^{-1}) \cdot C \cdot [I] \cdot \exp(\tau[K_i][M_{ii}]^{-1}) \cdot \exp(-\tau[K_i][M_{ii}]^{-1}) d\tau, \] (17)

where: \( LA = \ln(\{I\} \cdot [M_{ii}]^{-1} + C \cdot [I]) \).

Now we can disentangle exponents in our formula and integrate it:

\[ R = \tau \cdot [M_{ii}]^{-1} \cdot \Delta \cdot \left( \exp(-\tau[K_i][M_{ii}]^{-1} + LA) - C \cdot \exp(-\tau[K_i][M_{ii}]^{-1}) \right). \] (18)

V NUMERICAL EXAMPLES

The following pictures show the sensitivity term obtained for the node “1” and node “6” on one-dimensional model. FE-TS means the sensitivity obtained with finite element time stepping method (with 100 steps), Semi the result received with semi-discrete method and numerical time integration (12), and Semi-I the same, but using (18). We can observe good agreement of the results. The formula (18) covers the whole range of time.

Figure 7: Sensitivity term for node “1” 1D case. Figure 8: Sensitivity term for node “6” 1D case.

The comparison of results on two dimensional model one can see in Fig.9.

Figure 9: Sensitivity term for node “1” 2D case.

Then, the equation (18) was applied to evaluate sensitivity of the voltage induced in measurement coil of the NDE equipment. Three coils have been placed inside a long conducting pipe (Fig. 10). The coil in the middle is used for measurement, other two are excited differentially. Excitation has the form of a unit step impulse. The model exhibits cylindrical symmetry (Fig. 11) and can be analyzed using 2D axial-symmetric formulation.
Analysing the original and adjoint models, the values of measurement coil sensitivity versus conductivity in chosen elements were derived by means of (18).

VI CONCLUSIONS

Proposed method allows to evaluate the semi-discrete sensitivity of electromagnetic field quantities versus conductivity in finite element, without time-stepping. When using standard FE-TS method for the original and adjoint model (in backward time), practically constant time step needs to be applied. However, while solving inverse problems by means of the gradient method [1], the useful information is delivered not only by the first time steps, but also by the advanced time points. Then, we have to meet a compromise between the size of time step and the number of steps [2]. The aforementioned problem vanishes while using the semi-discrete time-domain sensitivity analysis. The drawback of this method are matrices, which are losing their symmetry and are no more banded. All calculations in this work were carried out with fully assigned matrices. Our experience shows, that despite of large memory demand the described method may be competitive versus finite elements with time stepping.

REFERENCES